

Suma de una serie finita de potencias

Dada la serie:

$$S = 1 + 2^p x + 3^p x^2 + \cdots + n^p x^{n-1}, \quad p > 0$$

Luego,

$$\begin{aligned} S &= \sum_{k=0}^{n-1} (k+1)^p x^k \\ S - xS &= \sum_{k=0}^{n-1} (k+1)^p x^k - \sum_{k=0}^{n-1} (k+1)^p x^{k+1} \\ &= \sum_{k=0}^{n-1} (k+1)^p x^k - \left(\sum_{k=0}^{n-2} (k+1)^p x^{k+1} + n^p x^n \right) \\ &= \sum_{k=0}^{n-1} (k+1)^p x^k - \left(\sum_{k=1}^{n-1} k^p x^k + n^p x^n \right) \\ &= \left(\sum_{k=0}^{n-1} (k+1)^p x^k - \sum_{k=0}^{n-1} k^p x^k \right) - n^p x^n \\ &= \left(\sum_{k=0}^{n-1} (k+1)^p - k^p \right) x^k - n^p x^n \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=0}^{p-1} \binom{p}{j} k^j \right) x^k - n^p x^n \\ (1-x)S &= \sum_{j=0}^{p-1} \binom{p}{j} \left(\sum_{k=0}^{n-1} k^j x^k \right) - n^p x^n \end{aligned}$$

Observación:

$$\sum_{k=0}^n \left(\sum_{j=0}^p v_{p,j} w_{k,j} \right) u_k = \sum_{j=0}^p v_{p,j} \left(\sum_{k=0}^n u_k w_{k,j} \right)$$

Por consiguiente,

$$S = \frac{1}{1-x} \left(\sum_{j=0}^{p-1} \binom{p}{j} \left(\sum_{k=0}^{n-1} k^j x^k \right) - n^p x^n \right)$$

Donde,

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$$